18 Find $\frac{d}{dx}\left(x\sin 2x\right)$, and hence find $\int_0^{\frac{\pi}{4}} x\cos 2x\,dx$.

19 **a** Show that $\frac{d}{dx}(\tan^3 x) = 3(\sec^4 x - \sec^2 x)$.

b Hence find $\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$.

ENRICHMENT

20 **a** Show that $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$.

b Hence find:

i $\int_0^{\frac{\pi}{2}} 2\sin 3x\cos 2x\,dx$

ii $\int_0^{\pi} \sin 3x\cos 4x\,dx$

c Show that $\int_{-\pi}^{\pi} \sin mx\cos nx\,dx = 0$ for positive integers m and n:

i using the primitive,

ii using symmetry arguments.

21 **a** Find the values of A and B in the identity

$$A(2\sin x + \cos x) + B(2\cos x - \sin x) = 7\sin x + 11\cos x.$$

b Hence show that $\int_0^{\frac{\pi}{2}} \frac{7\sin x + 11\cos x}{2\sin x + \cos x}\,dx = \frac{1}{2}(5\pi + 6\ln 2)$.

22 [The power series for sin x and cos x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .]$$

a We know that $\cos t \le 1$, for t positive. Integrate this inequality over the interval $0 \le t \le x$, where x is positive, and hence show that $\sin x \le x$.

b Change the variable to t, integrate the inequality $\sin t \le t$ over $0 \le t \le x$, and hence show that $\cos x \ge 1 - \frac{x^2}{2!}$.

c Do it twice more, and show that:

i $\sin x \ge x - \frac{x^3}{3!}$

ii $\cos x \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

d Now use induction (informally) to show that for all positive integers n,

$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \le \sin x + \frac{x^{4n+3}}{(4n+3)!},$$

and use this inequality to conclude that $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ converges, with limit sin x.

e Proceeding similarly, prove that $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ converges, with limit cos x.

f Use evenness and oddness to extend the results of (d) and (e) to negative values of x.