Exercise 7D

1 Find the following indefinite integrals.

a $\int \sec^2 x\, dx$ **b** $\int \cos x\, dx$ **c** $\int \sin x\, dx$

d $\int -\sin x\, dx$ **e** $\int 2\cos x\, dx$ **f** $\int \cos 2x\, dx$

g $\int \frac{1}{2}\cos x\, dx$ **h** $\int \cos \frac{1}{2}x\, dx$ **i** $\int \sin 2x\, dx$

j $\int \sec^2 5x\, dx$ **k** $\int \cos 3x\, dx$ **l** $\int \sec^2 \frac{1}{3}x\, dx$

m $\int \sin \frac{x}{2}\, dx$ **n** $\int -\cos \frac{1}{5}x\, dx$ **o** $\int -4\sin 2x\, dx$

p $\int \frac{1}{4}\sin \frac{1}{4}x\, dx$ **q** $\int 12\sec^2 \frac{1}{3}x\, dx$ **r** $\int 2\cos \frac{x}{3}\, dx$

2 Find the value of:

a $\int_0^{\frac{\pi}{2}} \cos x\, dx$ **b** $\int_0^{\frac{\pi}{6}} \cos x\, dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\, dx$

d $\int_0^{\frac{\pi}{3}} \sec^2 x\, dx$ **e** $\int_0^{\frac{\pi}{4}} 2\cos 2x\, dx$ **f** $\int_0^{\frac{\pi}{3}} \sin 2x\, dx$

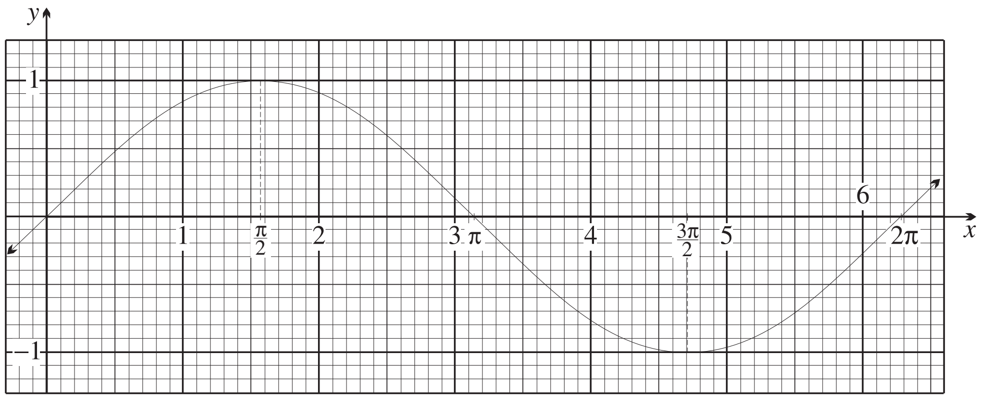
g $\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{1}{2}x\right) dx$ **h** $\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{1}{2}x\right) dx$ **i** $\int_0^{\pi} (2\sin x - \sin 2x)\, dx$

3 **a** The gradient function of a certain curve is given by $\frac{dy}{dx} = \sin x$. If the curve passes through the origin, find its equation.

b Another curve passing through the origin has gradient function $y' = \cos x - 2\sin 2x$. Find its equation.

c If $\frac{dy}{dx} = \sin x + \cos x$, and $y = -2$ when $x = \pi$, find y as a function of x.

4



The graph of $y = \sin x$ is sketched above.

a The first worked exercise in the notes for this section proved that $\int_0^{\pi} \sin x \, dx = 2$. Count squares on the graph of $y = \sin x$ above to confirm this result.

b On the same graph of $y = \sin x$, count squares and use symmetry to find:

i $\int_0^{\frac{\pi}{4}} \sin x \, dx$ **ii** $\int_0^{\frac{\pi}{2}} \sin x \, dx$ **iii** $\int_0^{\frac{3\pi}{4}} \sin x \, dx$

iv $\int_0^{\frac{5\pi}{4}} \sin x \, dx$ **v** $\int_0^{\frac{3\pi}{2}} \sin x \, dx$ **vi** $\int_0^{\frac{7\pi}{4}} \sin x \, dx$

c Evaluate these integrals using the fact that $-\cos x$ is a primitive of $\sin x$, and confirm the results of part **b**.

5 [Technology]

Programs that sketch the graph and then approximate definite integrals would help reinforce the previous very important investigation. The investigation could then be continued past $x = \pi$, after which the definite integral decreases again.

Similar investigation with the graphs of $\cos x$ and $\sec^2 x$ would also be helpful, comparing the results of computer integration with the exact results obtained by integration using the standard primitives.

6 Find the following indefinite integrals.

a $\int \cos(x + 2) \, dx$ **b** $\int \cos(2x + 1) \, dx$ **c** $\int \sin(x + 2) \, dx$

d $\int \sin(2x + 1) \, dx$ **e** $\int \cos(3x - 2) \, dx$ **f** $\int \sin(7 - 5x) \, dx$

g $\int \sec^2(4 - x) \, dx$ **h** $\int \sec^2\left(\frac{1-x}{3}\right) dx$ **i** $\int \sin\left(\frac{1-x}{3}\right) dx$

7 **a** Find $\int \left(6 \cos 3x - 4 \sin \tfrac{1}{2}x\right) dx$.

b Find $\int \left(8 \sec^2 2x - 10 \cos \tfrac{1}{4}x + 12 \sin \tfrac{1}{3}x\right) dx$.

8 **a** If $f'(x) = \pi \cos \pi x$ and $f(0) = 0$, find $f(x)$ and $f\left(\frac{1}{3}\right)$.

b If $f'(x) = \cos \pi x$ and $f(0) = \frac{1}{2\pi}$, find $f(x)$ and $f\left(\frac{1}{6}\right)$.

c If $f''(x) = 18 \cos 3x$ and $f'(0) = f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$.

DEVELOPMENT

9 Find the following indefinite integrals, where a, b, u and v are constants.

a $\int a \sin(ax + b) \, dx$ **b** $\int \pi^2 \cos \pi x \, dx$

c $\int \frac{1}{u} \sec^2(v + ux) \, dx$ **d** $\int \frac{a}{\cos^2 ax} \, dx$

10 a Copy and complete $1 + \tan^2 x = \ldots$, and hence find $\int \tan^2 x\, dx$.

b Simplify $1 - \sin^2 x$, and hence find the value of $\int_0^{\frac{\pi}{3}} \frac{2}{1 - \sin^2 x}\, dx$.

11 a Copy and complete $\int \frac{f'(x)}{f(x)}\, dx = \ldots$.

b Hence show that $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + \sin x}\, dx \doteqdot 0.4$.

12 a Use the fact that $\tan x = \frac{\sin x}{\cos x}$ to show that $\int_0^{\frac{\pi}{4}} \tan x\, dx = \frac{1}{2}\ln 2$.

b Use the fact that $\cot x = \frac{\cos x}{\sin x}$ to find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\, dx$.

13 a Find $\frac{d}{dx}(\sin^5 x)$, and hence find $\int \sin^4 x \cos x\, dx$.

b Find $\frac{d}{dx}(\tan^3 x)$, and hence find $\int \tan^2 x \sec^2 x\, dx$.

14 a Differentiate $e^{\sin x}$, and hence find the value of $\int_0^{\frac{\pi}{2}} \cos x e^{\sin x}\, dx$.

b Differentiate $e^{\tan x}$, and hence find the value of $\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x}\, dx$.

15 a Show that $\frac{d}{dx}(\sin x - x\cos x) = x \sin x$, and hence find $\int_0^{\frac{\pi}{2}} x \sin x\, dx$.

b Show that $\frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right) = \sin^3 x$, and hence find $\int_0^{\frac{\pi}{3}} \sin^3 x\, dx$.

16 Use the reverse chain rule $\int f'(x)\,(f(x))^n dx = \frac{(f(x))^{n+1}}{n + 1}$, to evaluate:

a $\int_0^{\pi} \sin x \cos^8 x\, dx$

b $\int_0^{\frac{\pi}{2}} \sin x \cos^n x\, dx$

c $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin^7 x\, dx$

d $\int_0^{\frac{\pi}{6}} \cos x \sin^n x\, dx$

e $\int_0^{\frac{\pi}{3}} \sec^2 x \tan^7 x\, dx$

f $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^n x\, dx$

17 a Show, by finding the integral in two different ways, that for constants C and D,

$$\int \sin x \cos x\, dx = \tfrac{1}{2}\sin^2 x + C = -\tfrac{1}{4}\cos 2x + D.$$

b How may the two answers be reconciled?

18 Find $\frac{d}{dx}\left(x \sin 2x\right)$, and hence find $\int_0^{\frac{\pi}{4}} x \cos 2x \, dx$.

19 **a** Show that $\frac{d}{dx}(\tan^3 x) = 3(\sec^4 x - \sec^2 x)$. **b** Hence find $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$.

ENRICHMENT

20 **a** Show that $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$.

b Hence find:

i $\int_0^{\frac{\pi}{2}} 2 \sin 3x \cos 2x \, dx$ **ii** $\int_0^{\pi} \sin 3x \cos 4x \, dx$

c Show that $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$ for positive integers m and n:

i using the primitive, **ii** using symmetry arguments.

21 **a** Find the values of A and B in the identity

$$A(2 \sin x + \cos x) + B(2 \cos x - \sin x) = 7 \sin x + 11 \cos x.$$

b Hence show that $\int_0^{\frac{\pi}{2}} \frac{7\sin x + 11\cos x}{2\sin x + \cos x} dx = \frac{1}{2}(5\pi + 6 \ln 2)$.

22 [The power series for sin x and cos x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.]$$

a We know that $\cos t \le 1$, for t positive. Integrate this inequality over the interval $0 \le t \le x$, where x is positive, and hence show that $\sin x \le x$.

b Change the variable to t, integrate the inequality $\sin t \le t$ over $0 \le t \le x$, and hence show that $\cos x \ge 1 - \frac{x^2}{2!}$.

c Do it twice more, and show that:

i $\sin x \ge x - \frac{x^3}{3!}$ **ii** $\cos x \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

d Now use induction (informally) to show that for all positive integers n,

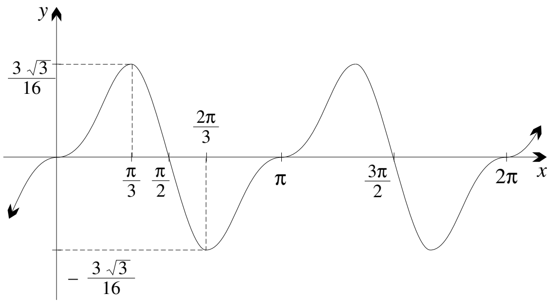
$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \le \sin x + \frac{x^{4n+3}}{(4n+3)!},$$

and use this inequality to conclude that $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ converges, with limit sin x.

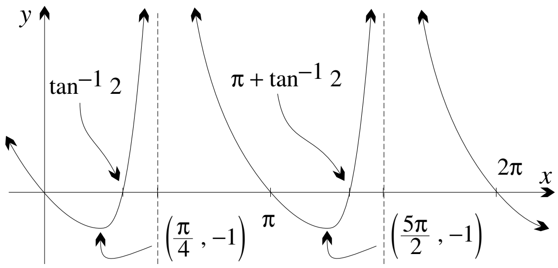
e Proceeding similarly, prove that $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ converges, with limit cos x.

f Use evenness and oddness to extend the results of (d) and (e) to negative values of x.

b maximum turning points $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{16}\right)$, $\left(\frac{4\pi}{3}, \frac{3\sqrt{3}}{16}\right)$,
minimum turning points $\left(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{16}\right)$, $\left(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{16}\right)$
horizontal points of inflexion $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$



c minimum turning points $\left(\frac{\pi}{4}, -1\right)$, $\left(\frac{5\pi}{4}, -1\right)$, vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, x-intercepts 0, π, 2π, $\tan^{-1} 2 \doteqdot 1.1$, $\pi + \tan^{-1} 2 \doteqdot 4.25$

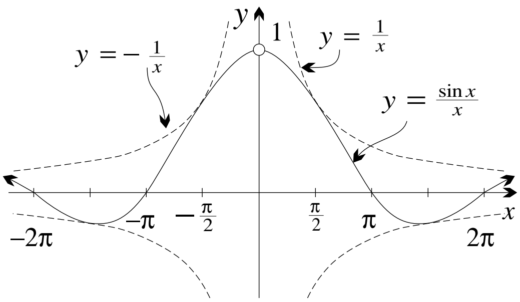


21 a Domain: $x \neq 0$, $f(x)$ is even because it is the ratio of two odd functions, the zeroes are $x = n\pi$ where n is an integer, $\lim_{x \to \infty} f(x) = 0$.

b $f'(x) = \frac{x\cos x - \sin x}{x^2}$, which is zero when $\tan x = x$.

c The graph of $y = x$ crosses the graph of $y = \tan x$ just to the left of $x = \frac{3\pi}{2}$, of $x = \frac{5\pi}{2}$ and of $x = \frac{7\pi}{2}$. Using the calculator, the three turning points of $y = f(x)$ are approximately $(1.43\pi, -0.217)$, $(2.46\pi, 0.128)$ and $(3.47\pi, -0.091)$.

d There is an open circle at $(0, 1)$.



Exercise 7D

1 a $\tan x + C$ **b** $\sin x + C$
c $-\cos x + C$ **d** $\cos x + C$
e $2\sin x + C$ **f** $\frac{1}{2}\sin 2x + C$
g $\frac{1}{2}\sin x + C$ **h** $2\sin\frac{1}{2}x + C$
i $-\frac{1}{2}\cos 2x + C$ **j** $\frac{1}{5}\tan 5x + C$
k $\frac{1}{3}\sin 3x + C$ **l** $3\tan\frac{1}{3}x + C$
m $-2\cos\frac{x}{2} + C$ **n** $-5\sin\frac{1}{5}x + C$
o $2\cos 2x + C$ **p** $-\cos\frac{1}{4}x + C$
q $-36\tan\frac{1}{3}x + C$ **r** $6\sin\frac{x}{3} + C$

2 a 1 **b** $\frac{1}{2}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\sqrt{3}$ **e** 1
f $\frac{3}{4}$ **g** 2 **h** 1 **i** 4

3 a $y = 1 - \cos x$
b $y = \sin x + \cos 2x - 1$
c $y = -\cos x + \sin x - 3$

6 a $\sin(x + 2) + C$ **b** $\frac{1}{2}\sin(2x + 1) + C$
c $-\cos(x + 2) + C$ **d** $-\frac{1}{2}\cos(2x + 1) + C$
e $\frac{1}{3}\sin(3x - 2) + C$ **f** $\frac{1}{5}\cos(7 - 5x) + C$
g $-\tan(4 - x) + C$ **h** $-3\tan\left(\frac{1 - x}{3}\right) + C$
i $3\cos\left(\frac{1 - x}{3}\right) + C$

7 a $2\sin 3x + 8\cos\frac{1}{2}x + C$
b $4\tan 2x - 40\sin\frac{1}{4}x - 36\cos\frac{1}{3}x + C$

8 a $f(x) = \sin\pi x$, $f\left(\frac{1}{3}\right) = \frac{1}{2}\sqrt{3}$
b $f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sin\pi x$, $f\left(\frac{1}{6}\right) = \frac{1}{\pi}$
c $f(x) = -2\cos 3x + x + \left(1 - \frac{\pi}{2}\right)$

9 a $-\cos(ax + b) + C$ **b** $\pi\sin\pi x + C$
c $\frac{1}{u^2}\tan(v + ux) + C$ **d** $\tan ax + C$

10 a $1 + \tan^2 x = \sec^2 x$, $\tan x - x + C$
b $1 - \sin^2 x = \cos^2 x$, $2\sqrt{3}$

11 a $\log_e f(x) + C$

12 a $\int \tan x = -\ln\cos x + C$
b $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx = \left[\log\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \log 2$

13 a $5\sin^4 x\cos x$, $\frac{1}{5}\sin^5 x + C$
b $3\tan^2 x\sec^2 x$, $\frac{1}{3}\tan^3 x + C$

14 a $\cos x e^{\sin x}$, $e - 1$ **b** $e^{\tan x} + C$, $e - 1$

15 a 1 **b** $\frac{5}{24}$

16 a $\frac{2}{9}$ **b** $\frac{1}{n + 1}$ **c** 0
d $\frac{1}{2^{n+1}(n + 1)}$ **e** $10\frac{1}{8}$ **f** $\frac{1}{n + 1}$

17 b $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, so $\frac{1}{2}\sin^2 x + C$
$= \frac{1}{4} - \frac{1}{4}\cos 2x + C = -\frac{1}{4}\cos 2x + \left(C + \frac{1}{4}\right)$
$= -\frac{1}{4}\cos 2x + D$, where $D = C + \frac{1}{4}$.

18 $\sin 2x + 2x\cos 2x$, $\frac{\pi - 2}{8}$

19 b $\frac{4}{3}$

20 b i $\frac{6}{5}$ **ii** $-\frac{6}{7}$

21 a $A = 5, B = 3$

Exercise 7E

1 a 1 square unit **b** $\frac{1}{2}$ square unit
2 a 1 square unit **b** $\sqrt{3}$ square units