11. INTEGRAL CALCULUS

• INTEGRATION

1. $\int x^n dx$
2. $\int (3x-5)(x+1)\, dx$
3. $\int \frac{1}{x} dx$
4. $\int \frac{1}{x\sqrt{x}} dx$
5. $\int \frac{dx}{\sqrt{x+1}}$
6. $\int \frac{x+1}{\sqrt{x}} dx$
7. $\int (ax+b)^n dx$
8. $\int 2\, dx$
9. $\int \frac{3x-5}{2} dx$
10. $\int \frac{2}{3x-5} dx$
11. $\int \frac{3x^2+2x}{x} dx$
12. $\int \frac{x}{x^2+1} dx$
13. $\int \frac{5x^2-3x+1}{\sqrt{x}} dx$
14. $\int (3x-1)^5 dx$
15. $\int e^x dx$
16. $\int \frac{x}{e} + \frac{e}{x} dx$
17. $\int \sqrt{e^x} dx$
18. $\int e^{ax+b} dx$
19. $\int \frac{e^x}{e^x+1} dx$
20. $\int \frac{e^x+1}{e^x} dx$

21. $\int \frac{e^{2x}+e^{x}}{e^{4x}}dx$

22. $\int \frac{f'(x)}{f(x)}dx$

23. $\int \frac{1}{ax+b}dx$

24. $\int \frac{3}{1+2x}dx$

25. $\int \frac{x}{1-x^2}dx$

26. $\int \frac{4x^2}{1+x^3}dx$

27. $\int 3^x dx$

28. $\int 3^{2x+1} dx$

29. $\int_{-2}^{2} 3x^3 - x\,dx$

30. $\int_{2}^{4} \frac{x^2}{x^3-1}dx$

- **APPLICATION OF THE PRIMITIVE FUNCTION**

1. Find the equation of the curve if

   a) $\frac{dy}{dx} = 4x + 1$ and the curve passes through the point $(0, 3)$
   b) $\frac{dy}{dx} = 1 - x^2$ and the curve passes through the point $(-3, 1)$
   c) $\frac{dy}{dx} = 2x + c$ and the curve has a minimum at the point $(2, -1)$
   d) $\frac{d^2y}{dx^2} = 2$ and the curve has a minimum at the point $(2, 5)$

- **DIFFERENTIATE AND HENCE INTEGRATE**

1. Find $\frac{d}{dx}(x^2-1)^5$ and hence $\int x(x^2-1)^4\,dx$
2. Find $\frac{d}{dx}(x^2-7)^4$ and hence $\int 5x(x^2-7)^3 dx$
3. Find $\frac{d}{dx}(x^3-3x)^{10}$ and hence $\int (x^2-1)(x^3-3x)^9 dx$
4. Find $\frac{d}{dx}\sqrt{3x^2+4}$ and hence $\int \frac{x}{\sqrt{3x^2+4}}dx$
5. Find $\frac{d}{dx}(sin^3x)$ and hence $\int sin^2x\, cosdx$
6. Find $\frac{d}{dx}(tan^3x)$ and hence $\int sec^2x\, tan^2x\, dx$
7. Find $\frac{d}{dx}(xe^{3x})$ and hence $\int xe^{3x}\, dx$

8. Find $\frac{d}{dx}\left(\ln\left(\frac{2+x}{2-x}\right)\right)$ and hence $\int \frac{1}{4-x^2}\,dx$

9. Find $\frac{d}{dx}\ln(\cos x)$ and hence $\int_0^{\frac{\pi}{3}} \tan x\,dx$

- **AREA AND VOLUME**

1.

a) Write down the formula for the area bounded between the curve $y = f(x)$ and lines $x = a$ and $x = b$

b) Write down the formula for the area bounded between the curve $x = g(y)$ and lines $y = c$ and $y = d$

c) Write down the area bounded between the 2 curves $y = f(x)$ and $y = g(x)$ if the curves intersect at $x = a$ and $x = b$

d) Write down the formula for the volume generated when the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is revolved about the x axis.

e) Write down the formula for the volume generated when the area between the curve $y = f(x)$ and the y axis is revolved about the y axis.

2. For the following questions write down how you would evaluate the areas of the regions without performing the integration. However, a sketch is required.

a) Find the area of the region bounded by the curve

$f(x) = x^2 - 9x + 14$, the x axis and the ordinates $x = 3$ and $x = 6$

b) Find the area of the region bounded by the curve

$f(x) = x^2 - 6x$ the x axis and the ordinates $x = 2$ and $x = 8$

c) Find the area of the region bounded by the x axis, the curve

$f(x) = (2x+1)^2$ the line $y = 8x$ given that their point of intersection is $(\frac{1}{2}, 4)$

d) Find the points where the line $y = 3x - 4$ intersects the parabola

$y = x^2 - 3x - 4$ and find the exact area bounded by the line and the parabola.

e) Find the area enclosed by the arc of the curve $x = y^2 - y$ and the y axis.

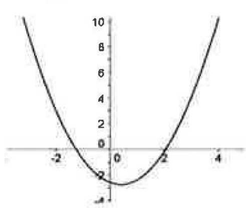
f) Find the area enclosed between the curve $y = \sqrt{x}$, the y axis and the line $y = 4$

g) Find the area of the region bounded by the curve $y = x^3$ the line $y = 8$ and the y axis

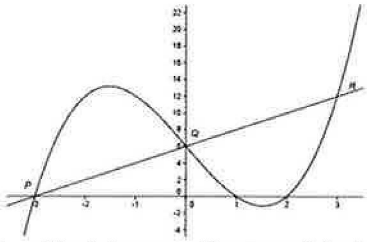
(do this in 2 different ways)

h) Write down an expression for the area bounded by the x axis and the curve

$y = x^2 - x - 2 = 0$ between $0 \le x \le 4$.

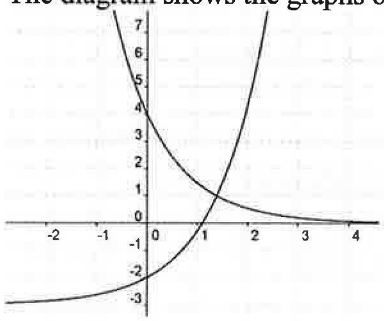


3. The graphs of $y = x^3 - 7x + 6$ and $y = 2x + 6$ are drawn



a) Find the coordinates of their points of intersection P, Q and R.
b) Find the area bounded by $y = x^3 - 7x + 6$ and $y = 2x + 6$

4. The diagram shows the graphs of $y = 4e^{-x}$ and $y = e^{x} - 3$



a) Show that the curves intersect when $e^{2x} - 3e^{x} - 4 = 0$

b) Hence show, by making a suitable substitution, that the x-coordinate of the point of intersection of the curves is $x = ln4$

c) Find the exact area bounded by the curves and the y $axis.$

5. Find the volume of the solid formed when $y = \ln x$ is rotated about the y axis between $y = 0$ and 1.

6. Find the volume of the solid generated when the area between the curves $y = x^2$ and $y = (x-2)^2$ and the x axis is rotated about the x axis.

7. Find the volume of the solid formed when the region between the curves $y = x^2$ and $y = 8 - x^2$ are rotated about

a) the x axis
b) the y axis

8. Write down the formula for approximating $\int_a^b f(x)dx$

a) Trapezoidal rule with
   i) 3 function values
   ii) 5 function values

b) Simpson's rule
   i) 3 function values
   ii) 5 function values

9. Use Trapezoidal rule to find the

a) Area enclosed by the curve $y = \sqrt{4 - x^2}$ the x axis and the lines $x = 0$ and $x = 2$ with

i) 2 sub-intervals

ii) 4 sub-intervals

b) Evaluate $\int_0^2 \sqrt{4 - x^2}\, dx$

c) Find the percentage error when using 4 sub-intervals correct to 1 d.p.

10. Use Simpson's rule to find the

a) Area enclosed by the curve $y = x^2 + 1$ the x axis and the lines $x = 0$ and $x = 2$ with

i) 3 function values

ii) 5 function values

b) Area under the curve $y = lnx$ between $x = 1\ and\ x = 5$. Use the ordinates given

x	1	2	3	4	5
y	0	0.693	1.099	1.386	1.609

c) Volume of the solid of revolution when the above area (b) is rotated about the x axis. Just show your working. There is no need to evaluate your answers.

11. When using Simpson's rule or the trapezoidal rule explain the relationship between n strips and m function or y values.

12. Using the trapezoidal rule with 2 strips evaluate an approximation for $\int_0^2 2^{-x}\, dx$.

Find the exact value for this area correct to 3dp. Find the percentage error.

11. INTEGRATION

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
2. $\int (3x-5)(x+1)\,\boldsymbol{dx} = \int 3x^2 - 2x - 5\ dx = x^3 - x^2 - 5x + C$
3. $\int \frac{1}{x} dx = \ln x + C$
4. $\int x^{-3/2} dx = -2x^{-1/2} + C$
5. $\int (x+1)^{-1/2} dx = 2(x+1)^{1/2} + C$
6. $\int \frac{x+1}{\sqrt{x}} dx = \int \sqrt{x} + \frac{1}{\sqrt{x}} \boldsymbol{dx} = \frac{2}{3} x\sqrt{x} + 2\sqrt{x} + C$
7. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$
8. $\int 2\,dx = 2x + C$
9. $\int \frac{3x-5}{2} dx = \frac{3x^2}{4} - \frac{5x}{2} + C$
10. $\int \frac{2}{3x-5} dx = \frac{2}{3} \ln(3x-5) + C$
11. $\int \frac{3x^2+2x}{x} dx = \int 3x + 2\,dx = \frac{3x^2}{2} + 2x + C$
12. $\int \frac{x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1) + C$
13. $\int \frac{5x^2-3x+1}{\sqrt{x}} dx = 2x^{5/2} - 2x^{3/2} + 2x^{1/2} + C$
14. $\int (3x-1)^5 dx = \frac{1}{18}(3x-1)^6 + C$
15. $\int e^x dx = e^x + C$
16. $\int \frac{x}{e} + \frac{e}{x}\ dx = \frac{x^2}{2e} + e\ln x + C$
17. $\int \sqrt{e^x}\, dx = 2e^{x/2} + C$
18. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$
19. $\int \frac{e^x}{e^x+1} dx = \ln(e^x+1) + C$
20. $\int \frac{e^x+1}{e^x} dx = x - e^{-x} + C$
21. $\int \frac{e^{2x}+e^x}{e^{4x}} dx = -\frac{1}{2} e^{-2x} - \frac{1}{3} e^{-3x} + C$
22. $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$
23. $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(ax+b) + C$

24. $\int \frac{3}{1+2x}dx = \frac{3}{2}\int \frac{2}{1+2x}dx = \frac{3}{2}\ln(1+2x) + C$

25. $\int \frac{x}{1-x^2}dx = -\frac{1}{2}\ln(1-x^2) + C$

26. $\int \frac{4x^2}{1+x^3}dx = \frac{4}{3}\ln(1+x^3) + C$

27. $\int 3^x dx = \frac{3^x}{ln3} + c$

28. $\int 3^{2x+1}dx = \frac{3^{2x+1}}{2ln3} + C$

29. $\int_{-2}^{2} 3x^3 - x\,dx = 0$ note odd function so no calculation needed

30. $\int_{2}^{4} \frac{x^2}{x^3-1}dx = \frac{1}{3}\left[ln(x^3-1)\right]_2^4 = \frac{1}{3}(ln63 - ln7)$

$= \frac{1}{3}ln9 = \frac{2}{3}ln3$

- **DIFFERENTIATE AND HENCE INTEGRATE**

1. $\frac{d}{dx}(x^2-1)^5 = 10x(x^2-1)^4$
   hence $\int 10x(x^2-1)^4\,dx = (x^2-1)^5$
   so $\int x(x^2-1)^4\,dx = \frac{1}{10}(x^2-1)^5 + C$
2. $\frac{d}{dx}(x^2-7)^4 = 8x(x^2-7)^3$
   hence $\int 5x(x^2-7)^3 dx = \frac{5}{8}(x^2-7)^4 + C$
3. $\frac{d}{dx}(x^3-3x)^{10} = 10\left(3x^2-3\right)(x^3-3x)^9$
   hence $\int (x^2-1)(x^3-3x)^9 dx = \frac{1}{30}(x^3-3x)^{10} + C$
4. $\frac{d}{dx}\sqrt{3x^2+4} = \frac{6x}{2}(3x^2+4)^{-1/2}$
   hence $\int \frac{3x}{\sqrt{3x^2+4}}dx = \sqrt{3x^2+4} + C$
   $\int \frac{x}{\sqrt{3x^2+4}}dx = \frac{1}{3}\sqrt{3x^2+4} + C$
5. $\frac{d}{dx}(sin^3x) = 3sin^2xcosx$
   hence $\int 3sin^2x\,cosdx = sin^3x + C$
   $\int sin^2x\,cosdx = \frac{1}{3}sin^3x + K$
6. $\frac{d}{dx}(tan^3x) = 3tan^2xsec^2x$
   hence $\int sec^2x\,tan^2x\,dx = \frac{1}{3}tan^3x + C$
7. $\frac{d}{dx}(xe^{3x}) = 3xe^{3x} + e^{3x}$
   hence $\int 3xe^{3x} + e^{3x}\;dx = xe^{3x}$
   so $\int 3xe^{3x}dx = xe^{3x} - \int e^{3x}\,dx$
   $= xe^{3x} - \frac{1}{3}e^{3x} + C$

8. $\frac{d}{dx}\left(\ln\left(\frac{2+x}{2-x}\right)\right) = \frac{d}{dx}\{\ln(2+x) - \ln(2-x)\}$

$$= \frac{1}{2+x} - \frac{-1}{2-x} = \frac{2-x+(2+x)}{4-x^2} = \frac{4}{4-x^2}$$

hence $\int \frac{1}{4-x^2}dx = \frac{1}{4}\ln\left(\frac{2+x}{2-x}\right) + C$

9. $\frac{d}{dx}\ln(cosx) = \frac{-sinx}{cosx}$

hence $\int_0^{\frac{\pi}{3}} tanx\ dx = -[\ln(cosx)]_0^{\frac{\pi}{3}} = 0 - ln\frac{1}{2} = ln2$

- **AREA AND VOLUME**

1.

a) $A = \int_a^b f(x)dx = \int_a^b y\ dx$

b) $A = \int_c^d g(y)dy = \int_a^b x\ dy$

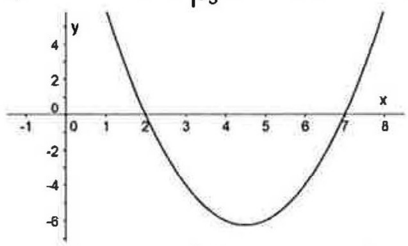
c) $A = \int_a^b f(x) - g(x)dx$

d) $V = \pi \int_a^b y^2\ dx$

e) $V = \pi \int_c^d x^2\ dy$

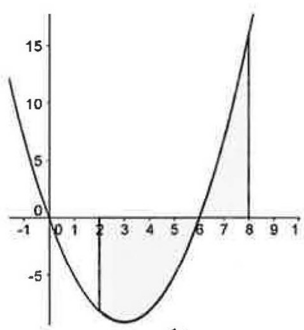
2.

a) $A = \left|\int_3^6 x^2 - 9x + 14\ dx\right|$

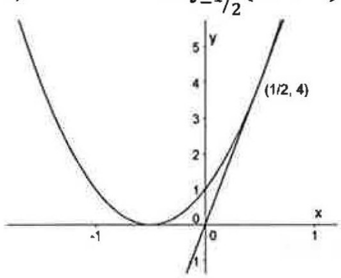


b) $A = \left|\int_2^6 x^2 - 6x\ dx\right| + \int_6^8 x^2 - 6x\ dx$

$$= \frac{80}{3} + \frac{44}{3} = 41\frac{1}{3}$$



c) $A = \int_{-1/2}^{1/2} (2x+1)^2\ dx - \frac{1}{2} \times \frac{1}{2} \times 4$

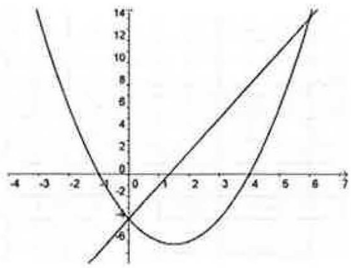


d)

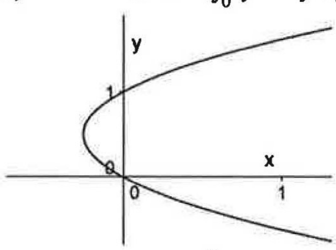
Solving simultaneously $3x - 4 = x^2 - 3x - 4$

$x^2 - 6x = 0$ so $x = 0, 6$

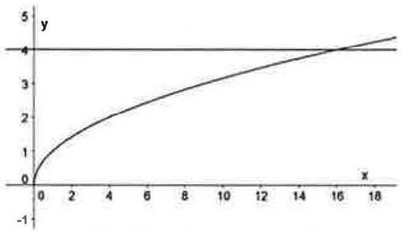
Area=$\int_0^6 3x - 4 - (x^2 - 3x - 4)dx$



e) $A = -\int_0^1 y^2 - y\,dy$ or $\left|\int_0^1 y^2 - y\,dy\right|$

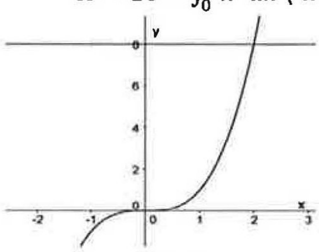


f) $A = \int_a^b x\,dy = \int_0^4 y^2\,dy$

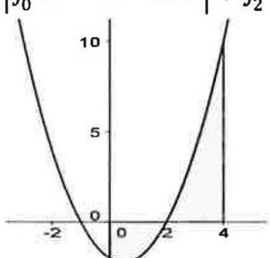


g) $(2,8)$ *is the point of intersection of the two curves*

$A = 16 - \int_0^2 x^3 dx$ (with x axis) or $A = \int_0^8 y^{\frac{1}{3}}\,dy$ (with the y axis)



h) $\left|\int_0^2 x^2 - x - 2dx\right| + \int_2^4 x^2 - x - 2dx$



3.

a) Solving equations simultaneously $x^3 - 7x + 6 = 2x + 6$

$x^3 - 9x = 0$
$x(x-3)(x+3) = 0$
$x = 0, 3, -3$
So $P(-3,0)$ $Q(0,6)$ $R(3,12)$

b) Area enclosed by 2 curves

$$= \int_{-3}^{0} x^3 - 7x + 6 - (2x+6)dx + \int_{0}^{3} 2x + 6 - (x^3 - 7x + 6)dx$$

$$= \int_{-3}^{0} x^3 - 9x\,dx + \int_{0}^{3} 9x - x^3 dx = \frac{81}{4} + \frac{81}{4} = \frac{81}{2}$$

4.

a) $4e^{-x} = e^x - 3$

$4 = e^{2x} - 3e^x$ and hence result.

b) $Let\ u = e^x \quad u^2 - 3u - 4 = 0$

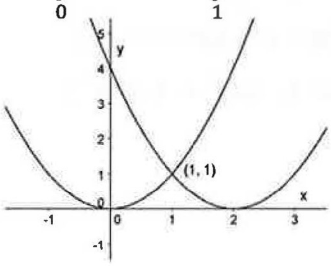
$(u-4)(u+1) = 0$
$u = 4, -1$
$e^x = 4\ or\ e^x = -1$ (no solution)
$x = ln4$

c) Area=$\int_0^{ln4} 4e^{-x} - (e^x - 3)dx = [-4e^{-x} - e^x + 3x]_0^{ln4}$

$$= -4e^{-ln4} - e^{ln4} + 3ln4 - (-4e^0 - e^0)$$
$$= -4 \times \frac{1}{4} - 4 + 3ln4 + 5 = 3ln4$$

5. $V = \pi \int_0^1 x^2\,dy = \pi \int_0^1 e^{2y}\,dy$
$$= \frac{\pi}{2}[e^{2y}]_0^1 = \frac{\pi}{2}(e^2 - 1)$$

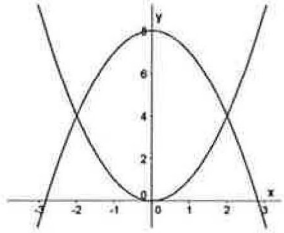
6. We have to find the sum of two volumes
$V = \pi \int_0^1 y^2\,dx + \pi \int_1^2 y^2\,dx$
$$= \pi \int_0^1 x^4\,dx + \pi \int_1^2 (x-2)^4\,dx = \frac{2\pi}{5}$$



7.

a) $V = \pi \int_{-2}^{2} (8 - x^2)^2 - x^4 dx = 170\frac{2}{3}$

b) $V = \pi \int_0^4 y\,dy + \pi \int_4^8 8 - y\,dy = 16$



8. where with n *subintervals each width h, so that* $h = \frac{b-a}{n}$

a)

i) $\int_a^b f(x)dx \cong \frac{h}{2}\{f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\}$

ii) $\int_a^b f(x)dx =\cong \frac{h}{2}\{f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(b)\}$

note $f(a) = f(x_0)$ and $f(b) = f(x_4)$

b)

i) $\int_a^b f(x)dx \cong \frac{h}{3}\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\}$ or

$\int_a^b f(x)dx \cong \frac{b-a}{6}\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\}$

ii) $\int_a^b f(x)dx \cong \frac{h}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\}$

9.

a)

i) $\int_0^2 \sqrt{4-x^2}\, dx \cong \frac{1}{2}[2 + 2\times\sqrt{3} + 0] = 2.732$

ii) $\int_0^2 \sqrt{4-x^2}\, dx \cong \frac{\frac{1}{2}}{2}\left[2 + 2\times\frac{\sqrt{15}}{2} + 2\times\sqrt{3} + 2\times\frac{\sqrt{7}}{2} + 0\right] = 2.996$

b) $\int_0^2 \sqrt{4-x^2}\, dx = \frac{1}{4}\pi r^2 = \pi \cong 3.14159$

c) percentage error=$\frac{0.1456}{3.1416} \times 100 = 4.6\%$

10.

d)

iii) $\int_0^2 f(x)dx \cong \frac{1}{3}(1 + 4\times 2 + 5) = 4\frac{2}{3}$

iv) $\int_0^2 f(x)dx \cong \frac{\frac{1}{2}}{3}(1 + 4\times 1.25 + 2\times 2 + 4\times 3.25 + 5) = 4\frac{2}{3}$

e) $\int_a^b f(x)dx \cong \frac{1}{3}(0 + 4\times 0.693 + 2\times 1.099 + 4\times 1.386 + 1.609) = 4.041$

f) $V = \pi\int y^2\, dx \cong \frac{\pi}{3}(1\times 0 + 4\times 0.693^2 + 2\times 1.099^2 + 4\times 1.386^2 + 1.069^2)$

11. n is one less than m

12. $\int_0^2 2^{-x}\, dx \cong \frac{1}{2}(2^0 + 2\times 2^{-1} + 2^{-2}) = 1.125$

Exact value= $\int_0^2 2^{-x}\, dx = \left[-\frac{2^{-x}}{ln2}\right]_0^2 = 1.082$

percentage error=$\frac{error}{exact} \times 100 = \frac{0.043}{1.082} \times 100 = 3.97\%$